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LETTER TO THE EDITOR

A q-Heisenberg algebra and a contact metric structure

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Abstract. The actions for two q-Heisenberg algebras are constructed by extending a contact metric structure of a Heisenberg group manifold to the corresponding line bundle. We obtain the q-Heisenberg algebras under a canonical quantization and regularization. Also the q parameter is shown to be the regularizing parameter.

Quantum groups have played a prominent role during recent years in the quantum integrable system, the rational conformal field theory, the knot theory, and so on.

A well known approach to finding them is the algebraic deformation of the Poisson Lie group [1, 2]. Thus it is said that the Lie algebra is quantized. Another approach is quantizing the space of functions on a Lie group, which is dual to a Lie algebra [3]. The two approaches are mutually dual [4]. In contrast, the Hall algebra [5] and endomorphisms on non-commutative coordinates [6] have been used.

Most approaches are restricted to a mathematical construction which is not started from a physical action and quantization. Thus we will find an action, conditions and a structure which give the quantum group.

The generators of a quantum group will be written as compositions of a q-Heisenberg algebra (or q-bosons) [7–9]. So we will concentrate on the q-Heisenberg algebra.

The actions for two q-Heisenberg algebras are constructed from extension of a contact metric structure of a Heisenberg group manifold to the corresponding line bundle. The q-Heisenberg algebras are obtained by a canonical quantization and regularization. The q parameter is shown to be the regularizing parameter.

The q-Heisenberg algebra is generated by $(a, a^\dagger, N)$ with a deformation parameter q satisfying

$$a^\dagger q^N = q^{-1} q^N a^\dagger \qquad a q^N = q q^N a \tag{1}$$

$$a a^\dagger - q^{\mp 1} a^\dagger a = q^{\pm N}. \tag{2}$$

The algebra becomes the Heisenberg algebra (bosons) in the limit $q \to 1$ as follows

$$a \to a_0 \qquad N = N_0 = a_0^\dagger a_0 \qquad [a_0, a_0^\dagger] = 1. \tag{3}$$

The algebra (3) is obtained by a canonical quantization of phase variables (p, x) of dimension two with the action one-form

$$A = \tfrac{1}{2}(p\, \mathrm{d}x - x\, \mathrm{d}p). \tag{4}$$

Extension to the higher dimensional phase space can be done similarly by introducing indices.

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To find a q-Heisenberg algebra, we take an action form related with the Heisenberg group manifold by extending (4). The manifold has three invariant one-forms as candidates of action one-form. Only one of them extends the A of (4) to three-dimensional phase space (p, x, θ)

$$\mathscr{A}=\tfrac{1}{2}(p\,\mathrm{d}x-x\,\mathrm{d}p)-\mathrm{d}\theta \tag{5}$$

with a topological constraint

$$\mathscr{A}\,\mathrm{d}\mathscr{A}\neq 0. \tag{6}$$

The action has three kinds of symmetries: a local $U(1)$ gauge symmetry, reparametrization and contact diffeomorphisms [10, 11]. Apparently, the $\mathscr{A}$ of (5) is equal to the A of (4) when the total derivative term $\mathrm{d}\theta$ is dropped by using a $U(1)$ gauge transformation. But we cannot take such a process in order to preserve the toplogical constraint (6). The $\mathscr{A}$ is called the canonical contact one-form [11].

Conversely, from the contact one-form we can reconstruct the Heisenberg group manifold by introducing a metric on a three space which is compatible with $\mathscr{A}$. The metric is well known as the Sasaki metric [12]. The A of (4) is considered as a $U(1)$ gauge field on the metric. The basis of the tangent space compatible with $\mathscr{A}$ forms the Heisenberg Lie algebra. We obtain the Heisenberg group manifold that has the $\mathscr{A}$ as one of its invariant one-forms.

At first, the Poisson bracket, related with (5), is expected to be obtained from a Kirrilov–Kostant symplectic structure on the coadjoint orbits of the group, and is equivalent to that form (4). But to fit the topological constraint (6), we take the Arnold form that has a new scaling variable $\lambda>0$ as a conjugate of θ [11]. The corresponding action one-form ω is defined on the four-dimensional phase space (p, x, λ, θ) as follows

$$\omega=\lambda\mathscr{A}=\frac{\lambda}{2}(p\,\mathrm{d}x-x\,\mathrm{d}p)-\lambda\,\mathrm{d}\theta. \tag{7}$$

In this case, λ is taken as a Lagrange multiplier of value $\lambda=1$. But the Arnold form is not suitable to define a Hilbert space after quantizing. We will comment briefly on this later.

We change this further by introducing complex variables (z, b) as

$$\begin{aligned}&\lambda=z^{\dagger}z \qquad \theta=\arg z\\ &b=\frac{1}{\sqrt{2}}(x+\mathrm{i}p) \qquad b^{\dagger}=\frac{1}{\sqrt{2}}(x-\mathrm{i}p).\end{aligned} \tag{8}$$

Here we consider the Heisenberg group manifold as a $U(1)$ principal bundle, and define the associated line bundle. The variable z represents the fibre coordinate of the line bundle [12, 13]. The invariant one-form (7) is modified into two kinds

$$\begin{aligned}\mathscr{L}_{+}&=\frac{1}{2\mathrm{i}}(zb\,\mathrm{d}(z^{\dagger}b^{\dagger})+z\,\mathrm{d}z^{\dagger}-(\leftrightarrow))\\ &=\lambda(\tfrac{1}{2}(p\,\mathrm{d}x-x\,\mathrm{d}p)-\mathrm{d}\theta)-\lambda H\,\mathrm{d}\theta\end{aligned} \tag{9}$$

$$\begin{aligned}\mathscr{L}_{-}&=\frac{1}{2\mathrm{i}}(z^{\dagger}b\,\mathrm{d}(zb^{\dagger})+z\,\mathrm{d}z^{\dagger}-(\leftrightarrow))\\ &=\lambda(\tfrac{1}{2}(p\,\mathrm{d}x-x\,\mathrm{d}p)-\mathrm{d}\theta)+\lambda H\,\mathrm{d}\theta.\end{aligned} \tag{10}$$

The symbol $(\leftrightarrow)$ means antisymmetrizing under exchange of variables, and H is the oscillator Hamiltonian

$$H = b^\dagger b = \tfrac{1}{2}(p^2 + x^2). \tag{11}$$

The modified canonical action one-form (9) and (10) are equivalent to the symplectic structure on the line bundle. A similar approach has been used to define a symplectic structure on a line bundle related to an expansion of [13]. It is interesting that the H in each second step is contained in the pure action-angle coordinates of each first step. The θ in each second step act like a time, so it is called the geometric time. The geometric (θ) and the external times are in the same direction on $\mathscr{L}_+$, but oppositely on $\mathscr{L}_-$. Thus we can expect two types of q-Heisenberg algebras related to the orientation of the group manifold, which are characterized by the geometric time θ.

Let us proceed to quantize by using $\mathscr{L}_+$ first. The canonical commutators are read from $\mathscr{L}_+$ (except (14)) as follows.

$$[z, z^\dagger] = 1 \tag{12}$$

$$[z, b] = [z^\dagger, b^\dagger] = 0 \tag{13}$$

$$[z, b^\dagger] = [z^\dagger, b] = 0 \tag{14}$$

$$[zb, z^\dagger b^\dagger] = 1. \tag{15}$$

For the Arnold form, (14) should be changed to

$$[z, b^\dagger] = b^\dagger \qquad [z^\dagger, b] = -b. \tag{16}$$

But the Arnold form does not show a Hilbert space for pure $b^\dagger$ and b. Thus we require (14).

The states for z and $z^\dagger$ are defined such that

$$z|0\rangle = 0 \qquad |n\rangle = \frac{1}{\sqrt{n!}}(z^\dagger)^n|0\rangle. \tag{17}$$

The state $|n\rangle$ is independent of b and $b^\dagger$ (from (13) and (14)). It is dependent on the geometric time (θ) and the Lagrangian multiplier (λ). So we should constrain them out by taking an expectation value, because the real dynamical variables are $b^\dagger$ and b. Thus the commutator (15), after taking the expectation for the fibre elements, will be an effective form.

But the expectation value for all states of $|n\rangle$ is divergent normally. We regularize it by introducing a parameter β such that

$$\sum_{n=0}^{\infty} \langle n|1|n\rangle = \sum_{n=0}^{\infty} 1 \to Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta n} = \frac{1}{1-e^{-\beta}}. \tag{18}$$

The $Z(\beta)$ is well known as the partition function or a $U(1)$ character. We can naturally expect the form in the theory itself by incorporating the oscillator Hamiltonian $h = z^\dagger z$ properly. The effective commutator of having the q parameter, which appears in [7], is followed

$$q^2 bb^\dagger - b^\dagger b = q^2 - 1 \tag{19}$$

$$q^2 = e^\beta. \tag{20}$$

Thus the first part relation of (2) can be seen by defining a and $a^\dagger$

$$\begin{aligned} a &= (q - q^{-1})^{-1/2} b q^{N/2} \\ a^\dagger &= (q - q^{-1})^{-1/2} q^{N/2} b^\dagger \end{aligned} \tag{21}$$

where

$$-\beta N=\ln(1-b^{\dagger}b). \tag{22}$$

As a result, we obtain the q-oscillator relating to the same time flow, and the q-parameter is interpreted as a regularization. The corresponding Hilbert space $|[n]\rangle$ $n=0,1,2,\ldots$ for b and $b^{\dagger}$ is defined

$$\begin{aligned}
&b|[0]\rangle=0\\
&b^{\dagger}|[n]\rangle=[n+1]^{1/2}|[n+1]\rangle\\
&b|[n]\rangle=[n]^{1/2}|[n-1]\rangle\\
&[n]=1-q^{-2n}.
\end{aligned} \tag{23}$$

Similarly we can perform quantization by using $\mathscr{L}_{-}$. The difference is in the commutator

$$[z^{\dagger}b, zb^{\dagger}]=1. \tag{24}$$

The q-deformation is obtained after similar steps

$$q^{-2}bb^{\dagger}-b^{\dagger}b=1-q^{-2}. \tag{25}$$

The equation is different from (19), replacing q by q^{-1}. We obtain the second part of relation (2) by defining

$$\begin{aligned}
&a=(q-q^{-1})^{-1/2}bq^{-N/2}\\
&a^{\dagger}=(q-q^{-1})^{-1/2}q^{-N/2}b^{\dagger}.
\end{aligned} \tag{26}$$

The corresponding Hilbert space is similarly defined.

As results, we construct two kinds of q-Heisenberg algebras related with the orientations of the geometric time (θ). Our formalism contains explicitly the regularizing parameter as a q deformation, which normally vanishes. But we expect that it gives a kind of physical meaning to the q-parameter.

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